Lecture 7

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JACOBI GEOMETRY I

(references: Chapter 1 of Tortorella's thesis, Vitagliano's work and sections 2.1, 2.3, 2.6 and 2.7 of Zapata's thesis)

Definition Proposition Exercise

(I) LOCAL LIE ALGEBRA REFRESHER

(reminder of lecture 6, see notes online for details and references)

local Lie algebra $(\Gamma(A), [\,,\,])$ $\mathbb{R}$-Lie bracket acting as a dif. op. (of order ≤ 1) on each of its arguments. $\mathrm{Diff}_1(A)$ fails to give examples.

derivative Lie algebra local Lie bracket acting as derivations

$[a, f\cdot b] = f\cdot[a,b] + \lambda_a[f]\cdot b \Leftrightarrow \sigma_{ad(a)} = \lambda_a \otimes id_A$, $\lambda : \Gamma(A) \to \Gamma(TM)$

symbol and squiggle $\Lambda^\#(df\otimes a) := \sigma_\lambda(df)(a)$, $\Lambda^\# : T^*M\otimes A \to TM$

Proposition 6.3 (Symbol-Squiggle Formula)

$(\Gamma(A), [\,,\,])$ derivative Lie algebra, by definitions of λ, Λ:

$$[f\cdot a, g\cdot b] = fg\cdot[a,b] + f\lambda_a[g]\cdot b - g\lambda_b[f]\cdot a + \Lambda^\#(df\otimes a)[g]\cdot b$$

satisfying

i) $\Lambda^\#(df\otimes a)[g]\cdot b = -\Lambda^\#(dg\otimes b)[f]\cdot a$ $\rightsquigarrow$ defines $\Lambda \in \Gamma(\Lambda^2(TM\otimes A^*)\otimes A)$

ii) $\lambda_{[a,b]} = [\lambda_a, \lambda_b]$

iii) $\lambda_{f\cdot a} = f\cdot\lambda_a + \Lambda^\#(df\otimes a)$

iv) $[\lambda_a, \Lambda^\#(df\otimes b)] = \Lambda^\#(d\lambda_a[f]\otimes b + df\otimes[a,b])$

v) $\Lambda(df\otimes a, d(\Lambda^\#(dg\otimes b)[h])\otimes c) + \text{cyclic} =$

$\lambda_b[f]\cdot\Lambda(dg\otimes a, dh\otimes c) + \text{cyclic}$

Proposition 6.3′ (Extension by Symbol)

Proposition 6.3' (Extension by Symbol)

$A \to M$ vector bundle, $\Gamma(A) = C^\infty(M)\cdot\Sigma$, datum of local Lie algebra equivalent:

1. a skewsymmetric $\mathbb{R}$-bilinear bracket $[\,,]: \Sigma\times\Sigma \to \Gamma(A)$
2. a $\mathbb{R}$-linear map $\lambda: \Gamma(A) \to \Gamma(TM)$
3. a $C^\infty(M)$-linear map $\Lambda^\#: \Gamma(T^*M\otimes A) \to \Gamma(TM)$

such that identities i)-v) in 6.3 hold.

anchor

Proposition 6.4 (Rank 1 Local Lie Algebras)

$rk(A) = 1 \Rightarrow (\Gamma(A), [\,,])$ local Lie algebra is a derivative Lie algebra

Proposition 6.5 (Rank ≥ 2 derivative Lie Algebras)

$rk(A) \geq 2 \Rightarrow (\Gamma(A), [\,,])$ derivative Lie algebra with anchor.

Motivates the identification of the category of Lie algebroids (lecture 3) and a category of local Lie algebras of rank 1 somehow.

II UNIT-FREE MECHANICS?

Consider a situation motivated by realistic physics: if observables of a classical system are represented by $\mathbb{R}$-valued smooth functions, where are the units? In practice, units are assumed fixed so that physical quantities are given by real numbers. The problem is motivated:

"Can we define a theory of mechanics in terms of unit-free observables on phase spaces?"

From the point of view of Kinematics we should identify a category of configuration spaces analogous to the category of smooth manifolds and a category of phase spaces analogous to the category of Poisson manifolds.

The answer to this question turns out to be rather direct and constructive once we move from representing scalars as elements of $\mathbb{R}$ to elements of arbitrary 1-dimensional vector spaces (without a choice of basis) and we adopt what I call the unit-free approach to line bundles (which is little more than an honest intrinsic/trivialisation-free treatment of line bundles, as in the work of Luca Vitagliano).

SUMMARY OF THE UNIT-FREE PHILOSOPHY

Unit-less	Unit-free
$\mathbb{R}$	L (1-dim vector space without choice of basis)
$\mathbb{R}$	Line (see III)
M	$L \to M$ (line bundle without choice of trivialisation)
Man	Line_{Man} (see IV)
$C^\infty(M)$	$\Gamma(L)$
Ring	Ring - Mod
TM	DL
T^*M	J^1L
$\varphi: M \to N$	$B_\varphi: L \to L'$
$T\varphi: TM \to TN$	$DB_\varphi: DL \to DL'$
$(M, \{,\})$ Poisson	$(L, \{,\})$ Jacobi (see lecture 8)

III THE CATEGORY OF LINES

The Category of Lines $\text{Line} := \left(\underbrace{\text{1-dimensional real vector spaces}}_{\text{lines } L}, \underbrace{\text{invertible linear maps}}_{\text{factors } B} \right)$

Choice of Unit $u \in L^\times := L \setminus \{0\}$, i.e. choice of basis, induces $L \cong \mathbb{R}$

The Category of Line-Vector Spaces $\text{LVect} := \text{Vect} \times \text{Line}$
factor
lines

$$\varphi^B : V^L \to W^L$$

linear map

vector spaces

Taking a "L-rooted" approach (avoiding the appearance of tensor powers of lines) we can define a structure that mirrors that of ordinary vector spaces:

L-Direct Sum $V^L \oplus W^L := (V \oplus W)^L$ **Subspaces and Quotients** $(U \subset V)^L, (V/U)^L$

L-Duality $V^{*L} := (V^* \otimes L)^L$, $\varphi^{*B} : W^{*L'} \to V^{*L}$ (using $B^{-1} : L' \to L$)

Write the explicit definition of φ^{*B} (hint: $V^* \otimes L = \mathrm{Hom}_{Vect}(V, L)$).

Show that $(V^{*L})^{*L} \cong V^L$ as LV spaces.

Define L-annihilator U^{0L} and show $U^{*L} \cong V^{*L}/U^{0L}$

L-tensors

$$\mathcal{T}^r_0(V^L) := (V \otimes \overset{r}{\cdots} \otimes V)^L = \mathcal{T}^r_0(V)^L, \quad \mathcal{T}^\bullet(V^L) := \bigoplus_{n=0}^{\infty} \mathcal{T}^n_0(V^L)$$

$$\mathcal{T}^0_q(V^L) := (V^* \otimes \cdots \otimes V^* \otimes L)^L = (\mathcal{T}^0_q(V) \otimes L)^L, \quad \mathcal{T}_\bullet(V^L) := \bigoplus_{n=0}^{\infty} \mathcal{T}^0_n(V^L)$$

We can define **L-pushforwards** in an obvious way $(\varphi^B)_* := (\varphi_*)^B$ and the definition of L-dual above φ^{*B} naturally generalises to the notion of **L-pullback** $\varphi^{*B} : \mathcal{T}^0_\bullet(W^{L'}) \to \mathcal{T}^0_\bullet(V^L)$.

These L-rooted definitions don't allow for a direct analogue of the tensor product as an associative operation between general tensors. In its place we find a $(\mathcal{T}(V), \otimes)$-module structure that we denote by the same symbol $\otimes$ in a silght abuse of notation:

Let $A \in \mathcal{T}^r_0(V)$, $\omega \in \mathcal{T}^0_q(V)$, $B \in \mathcal{T}^s_0(V^L)$, $\alpha \in \mathcal{T}^0_p(V^L)$

$$A \otimes B := A \otimes B \text{ (honest } \otimes)$$

$\omega \otimes \alpha$ defined by $\omega \otimes \alpha(v_1, \dots, v_q, w_1, \dots, w_p) := \omega(v_1, \dots, v_q) \cdot \alpha(w_1, \dots, w_p)$.

Note that our definition is such that $\mathcal{T}^\bullet(V^L)$ is just $\mathcal{T}^\bullet(V)$ "remembering" a choice of line L, however $\mathcal{T}_\bullet(V^L)$ is fundamentally different from $\mathcal{T}_\bullet(V)$ since they are the L-valued tensors on V. Note that, by convention:

$$\mathcal{T}^0(V^L) = \mathbb{R}, \quad \mathcal{T}_0(V^L) = L$$

Pushforwards became obvious module maps $\varphi^{*B}: \mathcal{T}^\bullet(V^L) \to \mathcal{T}^\bullet(W^{L'})$ and by setting $\varphi^{*B} = B^{-1}: L' \to L$ we find that so are pullbacks:

$$\alpha, \beta \in \mathcal{T}_\bullet(V^L) \quad \varphi^{*B}(\alpha+\beta) = \varphi^{*B}\alpha + \varphi^{*B}\beta, \quad \varphi^{*B}(\omega \otimes \alpha) = \varphi^{*}\omega \otimes \varphi^{*B}\alpha.$$
$$\omega \in \mathcal{T}_\bullet(V)$$

These constructions behave well under antisymmetrisation, which allows for the definition of L-multivectors and L-forms:

$$\wedge^\bullet(V^L) := (\wedge^\bullet(V))^L \qquad \wedge_\bullet(V^L) := (\wedge^\bullet(V^*) \otimes L)^L$$

which naturally carry $(\wedge^\bullet(V), \wedge)$- and $(\wedge^\bullet(V^*), \wedge)$-module structures respectively.

* Remark: the seemingly asymmetrical definition of L-tensors and the lack of an associative tensor product is directly related with our L-rooted approach, in which we actively avoid tensor powers of lines. If we allowed for tensor powers of lines, we would end up in the category of dimensioned structures (first identified by Dolan and Baez in the 2000s and studied in some detail in Chapter 7 of my thesis).

IV) THE CATEGORY OF LINE BUNDLES

The Category of Line Bundles

$$\mathrm{Line}_{\mathrm{Man}} := \left(\begin{array}{ll} \text{rank 1 vector bundles} & \text{fibre-wise invertible} \\ \text{over smooth manifolds}, & \text{vector bundle morphisms} \end{array} \right)$$

this is essentially the category of manifolds Man fibered with the category of lines Line. The following proposition shows that the usual object-centric constructions of manifolds (submanifolds, products and quotients) generalise to the category of line bundles:

Proposition 7.1 A submanifold $i: S \hookrightarrow M$ induces a line bundle L_S and an injective factor given by:

$$\begin{tikzcd} L_S := i^*L \arrow[r, "\iota"] \arrow[d] & L \arrow[d] \\ S \arrow[r, "i"'] & M \end{tikzcd}$$

Proposition 7.2 A Lie group acting via factors on a line bundle $G \curvearrowright L$, covering a free and proper action on the base manifold $G \curvearrowright M$ so that there is an orbit space projection $p: M \to M/G$, induces a line bundle L/G and a surjective factor:

$$\begin{tikzcd} L \arrow[r, "\pi"] \arrow[d] & L/G \arrow[d] \\ M \arrow[r, "p"'] & M/G \end{tikzcd}$$

Proposition 7.3 Given any two line bundles $\begin{smallmatrix} L_1 \\ \downarrow \\ M_1 \end{smallmatrix}$, $\begin{smallmatrix} L_2 \\ \downarrow \\ M_2 \end{smallmatrix}$ there is a canonical construction of the set of fibre-by-fibre isomorphisms:

$$M_1 \xleftarrow{p_1} M_1 \boxtimes M_2 \xrightarrow{p_2} M_2$$

which is shown to be locally $M_1 \times M_2 \times \mathbb{R}^\times$, called the base product, and a line bundle $L_1 \boxtimes L_2$ with canonical factors

$$\begin{tikzcd} L_1 \arrow[d] & L_1 \boxtimes L_2 \arrow[l, "P_1"'] \arrow[r, "P_2"] \arrow[d] & L_2 \arrow[d] \\ M_1 & M_1 \boxtimes M_2 \arrow[l, "p_1"] \arrow[r, "p_2"'] & M_2 \end{tikzcd}$$

called the line product, wich gives a categorical product $(\mathrm{Line}, \boxtimes)$.

Proofs. (See section 2.6.1 of my thesis.)

The fact that factors $b: L \to L'$ are fibre-wise invertible allows to define pullback of sections

$b: L \to L'$
$\varphi: \underset{M}{\downarrow} \to \underset{N}{\downarrow}$

$$b^*: \Gamma(L') \to \Gamma(L) \quad \text{via} \quad (b^*s)(x) := b_x^{-1}(s(\varphi(x))).$$

From this definition follows immediately that pullbacks interact with the $C^\infty(N)$-module structure of sections as follows:

$$f \in C^\infty(N), \qquad b^*(f \cdot s) = (\varphi^* f) \cdot b^* s.$$

This implies that the assignment of sections to line bundles becomes contravariant functor

$$\Gamma : \mathrm{Line}_{\mathrm{Man}} \longrightarrow \mathrm{RMod}$$

where RMod is the category of modules over rings with additive maps covering ring morphisms.

By fibering the category of manifolds Man with the category of LV spaces LVect we naturally obtain

The Category of Line-Vector Bundles

$$\mathrm{LVect}_{\mathrm{Man}} := \left(\begin{array}{l} \text{pairs of vector bundles} \\ \text{and line bundles over} \\ \text{the same manifold} \end{array} , \begin{array}{l} \text{pairs of vector bundle maphisms} \\ \text{and factors covering the same} \\ \text{smooth map between bases} \end{array} \right)$$

$$\begin{tikzcd} A^L \arrow[d] \\ M \end{tikzcd} \qquad \qquad \begin{tikzcd} A^L \arrow[r, "F^b"] \arrow[d] & B^{L'} \arrow[d] \\ M \arrow[r, "\varphi"'] & N \end{tikzcd}$$

By fixing a line bundle $\begin{smallmatrix} L \\ \downarrow \\ M \end{smallmatrix}$ we can reproduce all the definitions given in LVect shown in III above by working fibre-wise.

The presence of the categorical product of line bundles induces another on LV bundles: the direct sum of LV bundles is defined as:

$$A^L, B^{L'} \in \mathrm{LVect}_{\mathrm{Man}} \, , \quad A^L \boxplus B^{L'} := \left(p_1^* A \oplus_{M \times N} p_2^* B \right)^{L \boxtimes L'} .$$

Sections of LV bundles are formally regarded as $\Gamma(A^L) := (\Gamma(A), \Gamma(L))$ and are subject to the same subtleties regarding LV bundle morphisms as sections of regular vector bundles. In particular, a LV bundle morphism $F^b : A^L \to B^{L'}$ only defines the notion of F^b-relatedness:

$$\Gamma(A^L) \sim_{F}^{b} \Gamma(B^{L'})$$

in general, which becomes the push forward F^b_* when the base map is a difeomorphism and only the pullback F^{*b} for L-dual sections:

$$F^{*b} : \Gamma(B^{*L'}) \to \Gamma(A^{*L})$$

is defined in general. This map naturally extends to a morphism of modules for higher tensors:

$$F^{*b} : \Gamma(\mathcal{J}_\bullet(B^{L'})) \to \Gamma(\mathcal{J}_\bullet(A^L)).$$